The Annals of Statistics
2012, Vol. 40, No. 4, 2005–2013
DOI: 10.1214/12-AOS1020
Main article DOI: 10.1214/11-AOS949

REJOINDER: LATENT VARIABLE GRAPHICAL MODEL SELECTION VIA CONVEX OPTIMIZATION

By Venkat Chandrasekaran, Pablo A. Parrilo and Alan S. Willsky

California Institute of Technology, Massachusetts Institute of Technology and Massachusetts Institute of Technology

1. Introduction. We thank all the discussants for their careful reading of our paper, and for their insightful critiques. We would also like to thank the editors for organizing this discussion. Our paper contributes to the area of high-dimensional statistics which has received much attention over the past several years across the statistics, machine learning and signal processing communities. In this rejoinder we clarify and comment on some of the points raised in the discussions. Finally, we also remark on some interesting challenges that lie ahead in latent variable modeling.

Briefly, we considered the problem of latent variable graphical model selection in the Gaussian setting. Specifically, let X be a zero-mean Gaussian random vector in $\mathbb{R}^{p+h}$ with O and H representing disjoint subsets of indices in $\{1,\ldots,p+h\}$ with $|O|=p$ and $|H|=h$. Here the subvector X_O represents the observed variables and the subvector X_H represents the latent variables. Given samples of only the variables X_O, is it possible to consistently perform model selection? We noted that if the number of latent variables h is small relative to p and if the conditional statistics of the observed variables X_O conditioned on the latent variables X_H are given by a sparse graphical model, then the marginal concentration matrix of the observed variables X_O is given as the sum of a sparse matrix and a low-rank matrix. As a first step we investigated the identifiability of latent variable Gaussian graphical models—effectively, this question boils down to one of uniquely decomposing the sum of a sparse matrix and a low-rank matrix into the individual components. By studying the geometric properties of the algebraic varieties of sparse and low-rank matrices, we provided natural sufficient conditions for identifiability and gave statistical interpretations of these conditions. Second, we proposed the following regularized maximum-likelihood estimator to

Received May 2012.

decompose the concentration matrix into sparse and low-rank components:

$$(\hat{S}_n, \hat{L}_n) = \arg\min_{S,L} -\ell(S - L; \Sigma_O^n) + \lambda_n(\gamma\|S\|_1 + \operatorname{tr}(L))$$
$$\text{s.t. } S - L \succ 0, L \succeq 0. \tag{1.1}$$

Here Σ_O^n represents the sample covariance formed from n samples of the observed variables, ℓ is the Gaussian log-likelihood function, $\hat{S}_n$ represents the estimate of the conditional graphical model of the observed variables conditioned on the latent variables, and $\hat{L}_n$ represents the extra correlations induced due to marginalization over the latent variables. The ℓ_1 norm penalty induces sparsity in $\hat{S}_n$ and the trace norm penalty induces low-rank structure in $\hat{L}_n$. An important feature of this estimator is that it is given by a convex program that can be solved efficiently. Our final contribution was to establish the high-dimensional consistency of this estimator under suitable assumptions on the Fisher information underlying the true model (in the same spirit as irrepresentability conditions for sparse model selection [11, 16]).

2. Alternative estimators. A number of the commentaries described alternative formulations for estimators in the latent variable setting.

2.1. *EM-based methods.* The discussions by Yuan and by Lauritzen and Meinshausen describe an EM-based alternative in which the rank of the matrix L is explicitly constrained:

$$(\hat{S}_n, \hat{L}_n) = \arg\min_{S,L} -\ell(S - L; \Sigma_O^n) + \lambda_n\|S\|_1$$
$$\text{s.t. } S - L \succ 0, L \succeq 0, \operatorname{rank}(L) \leq \mathrm{r}. \tag{2.1}$$

The experimental results based on this approach seem quite promising, and certainly deserve further investigation. On the one hand, we should reiterate that the principal motivation for our convex optimization based formulation was to develop a method for latent variable modeling with provable statistical *and* computational guarantees. One of the main drawbacks of EM-based methods is the existence of local optima in the associated variational formulations, thus leading to potentially different solutions depending on the initial point. On the other hand, one of the reasons for the positive empirical behavior observed by Yuan and by Lauritzen and Meinshausen may be that all the local optima in the experimental settings considered by the authors may be "good" models. Such behavior has in fact been rigorously characterized recently for certain nonconvex estimators in some missing data problems [7].

One of the motivations for the EM proposal of Yuan and of Lauritzen and Meinshausen seems to be that there are fairly mature and efficient solvers for the graphical lasso. As our estimator is relatively newer and as its properties

are better understood going forward, we expect that more efficient solvers will be developed for (1.1) as well. Indeed, the LogdetPPA solver [15] that we cite in our paper already scales to instances involving several hundred variables, while more recent efforts [8] have resulted in algorithms that scale to instances with several thousand variables.

2.2. *Thresholding estimators.* Ren and Zhou propose and analyze an interesting thresholding based estimator for decomposing a concentration matrix into sparse and low-rank components. They apply a two-step procedure—ℓ_1 norm thresholding followed by trace norm thresholding—to obtain the sparse component followed by the low-rank component. Roughly speaking, this two-step estimator can be viewed as the application of the first cycle of a block coordinate descent procedure to compute our estimator that alternately updates the sparse and low-rank pieces (we also refer the reader to the remarks in [1]).

However, in Theorem 1 in the discussion by Ren and Zhou, a quite stringent assumption requires that in some scaling regimes the true low-rank component L^* must vanish, that is, $\|L^*\|_{\ell_\infty} \lesssim \sqrt{\frac{\log p}{n}} \to 0$. The reason for this condition is effectively to ensure sign consistency in recovering the sparse component. In a pure sparse model selection problem (with no low-rank component in the population), the deviation away from the sparse component is given only by noise due to finite samples and this deviation is on the order of $\sqrt{\frac{\log p}{n}}$ in the Gaussian setting—consequently, sparse model selection via ℓ_1 norm thresholding is sign-consistent when the minimum magnitude nonzero entry in the true model is larger than $\sqrt{\frac{\log p}{n}}$. In contrast, if the true model consists of both a sparse component and a low-rank component, the total deviation away from the sparse component in the finite sample regime is given by both sample noise as well as the low-rank component. This seems to be the reason for the stringent assumption on the vanishing of the low-rank component in Theorem 1 of Ren and Zhou.

More broadly, one of the motivations of Ren and Zhou in proposing and analyzing their estimator is that it may be possible to weaken the assumptions on the minimum magnitude nonzero entry θ of the true sparse component S^* and the minimum nonzero singular value σ of the true low-rank component L^*—whether this is possible under less stringent assumptions on L^* is an interesting question, and we comment on this point in Section 3 in the more general context of potentially improving the rates in our paper.

2.3. *Other proposals.* Giraud and Tsybakov propose two alternative estimators for decomposing a concentration matrix into sparse and low-rank components. While our approach (1.1) builds on the graphical lasso, their proposed approaches build on the Dantzig selector of Candès and Tao [2]

and the neighborhood selection approach of Meinshausen and Bühlmann [9]. Several comments are in order here.

First, we note that the extension of neighborhood selection proposed by Giraud and Tsybakov to deal with the low-rank component begins by reformulating the neighborhood selection procedure to obtain a "global" estimator that simultaneously estimates all the neighborhoods. This reformulation touches upon a fundamental aspect of latent variable modeling. In many applications marginalization over the latent variables typically induces correlations between most pairs of observed variables—consequently, local procedures that learn model structure one node at a time are ill-suited for latent variable modeling. Stated differently, requiring that a matrix be sparse with few nonzeros per row or column (e.g., expressing preference for a graphical model with bounded degree) can be done by imposing column-wise constraints. On the other hand, the constraint that a matrix be low-rank is really a global constraint expressed by requiring all minors of a certain size to vanish. Thus, any estimator for latent variable modeling (in the absence of additional conditions on the latent structure) must necessarily be global in nature.

Second, we believe that the reformulation based on the Dantzig selector perhaps ought to have an additional constraint. Recall that the Dantzig selector [2] constrains the ℓ_∞ norm (the dual norm of the ℓ_1 norm) of the correlated residuals rather than the ℓ_2 norm of the residuals as in the lasso. As the dual norm of our combined ℓ_1/trace norm regularizer involves both an ℓ_∞ norm and a spectral norm, the following constraint set may be more appropriate in the Dantzig selector based reformulation of Giraud and Tsybakov:

$$\mathcal{G} = \{(S, L) : \|\Sigma_O^n(S + L) - I\|_{\ell_\infty} \leq \gamma\lambda_n, \|\Sigma_O^n(S + L) - I\|_2 \leq \lambda_n\}.$$

Finally, we note that the Dantzig selector of [2] has the property that its constraint set contains the lasso solution (with the same choice of regularization/relaxation parameters). In contrast, this property is not shared in general by the Dantzig selector reformulation of Giraud and Tsybakov in relation to our regularized maximum-likelihood estimator (1.1). It is unclear how one might achieve this property via suitable convex constraints in a Dantzig selector type reformulation of our estimator.

In sum, both of these alternative estimators deserve further study.

3. Comments on rates. Several of the commentaries (Wainwright, Giraud and Tsybakov, Ren and Zhou and Candès and Soltanolkotabi) bring up the possibility of improving the rates given in our paper. At the outset we believe that $n \gtrsim p$ samples is inherent to the latent variable modeling problem if spectral norm consistency is desired in the low-rank component. This is to be expected since the spectral norm of the deviation of a sample

covariance from the underlying population covariance is on the order of $\sqrt{\frac{p}{n}}$. However, some more subtle issues remain.

Giraud and Tsybakov point out that one may be concerned purely with estimation of the sparse component, and that the low-rank component may be a "nuisance" parameter. While this is not appropriate in every application, in problem domains where the conditional graphical model structure of the observed variables is the main quantity of interest one can imagine quantifying deviations in the low-rank component via "weaker" norms than the spectral norm—this may lead to consistent estimates for the sparse component with $n \ll p$ samples. The analysis in our paper does not rule out this possibility, and a more careful investigation is needed to establish such results.

Ren and Zhou suggest that while $n \gtrsim p$ may be required for consistent estimation, one may be able to weaken the assumptions on θ and σ (the minimum magnitude nonzero entry of the sparse component and the minimum nonzero singular value of the low-rank component, respectively). From the literature on sparse model selection, a natural lower bound on the minimum magnitude nonzero entry for consistent model selection is typically given by the size of the noise measured in the ℓ_∞ norm (the dual of the ℓ_1 regularizer). Building on this intuition, a natural lower bound that one can expect in our setting on θ is $\frac{1}{\gamma}\|\Sigma_O^n - \Sigma\|_{\ell_\infty}$, while a natural bound on σ would be $\|\Sigma_O^n - \Sigma\|_2$. The reason for this suggestion is that $\max\{\frac{\|S\|_{\ell_\infty}}{\gamma}, \|L\|_2\}$ is the dual norm of the regularizer used in our paper. Therefore, it may be possible to only require $\theta \sim \frac{1}{\gamma}\sqrt{\frac{\log p}{n}}$ and $\sigma \sim \sqrt{\frac{p}{n}}$. However, one issue here is that the ℓ_∞ norm bound kicks in when $n \gtrsim \log p$ with probability approaching one polynomially fast, while the spectral norm bound only kicks in when $n \geq p$ but holds with probability approaching one exponentially fast. Thus (as also noted by Giraud and Tsybakov), it may be possible that $n \gtrsim p$ is required for overall consistent estimation, but that the assumption on θ could be weakened by only requiring that the probability of consistent estimation approach one polynomially fast.

Candès and Soltanolkotabi comment that it would be of interest to establish an "adaptivity" property whereby if no low-rank component were present, the number of samples required for consistent estimation would boil down to just the rate for sparse graphical model selection, that is, $n \sim \log p$. While such a feature would clearly be desirable to establish for our estimator, one potential roadblock may be that our estimator (1.1) "searches" over a larger classes of models than just those given by sparse graphical models; consequently, rejecting the hypothesis that the observed variables are affected by any latent variables may require that $n \gg \log p$. This question deserves further investigation and, as suggested by Candès and Soltanolkotabi,

recent results on adaptivity could inform a more refined analysis of our estimator.

Finally, Wainwright suggests the intriguing possibility that faster rates may be possible if the low-rank component has additional structure. For example, there may exist a sparse factorization of the low-rank component due to special structure between the latent and observed variables. In such settings the trace norm regularizer applied to the low-rank component is not necessarily the tightest convex penalty. In recent joint work by the authors and Recht [4], a general framework for constructing convex penalty functions based on some desired structure is presented. The trace norm penalty for inducing low-rank structure is motivated from the viewpoint that a low-rank matrix is the sum of a small number of rank-one matrices and, therefore, the norm induced by the convex hull of the rank-one matrices (suitably scaled) is a natural convex regularizer as this convex hull (the trace norm ball) carries precisely the kind of facial structure required for inducing low-rank structure in matrices. In this spirit, one can imagine constructing convex penalty functions by taking the convex hull of *sparse* rank-one matrices. While this convex hull is in general intractable to represent, relaxations of this set that are tighter than the trace norm ball could provide faster rates than can be obtained by using the trace norm.

4. Weakening of irrepresentability conditions. Wainwright asks a number of insightful questions regarding the potential for weakening our Fisher information based conditions. Giraud and Tsybakov also bring up connections between our conditions and irrepresentability conditions in previous papers on sparse model selection [11, 16].

In order to better understand if the Fisher information based conditions stated in our paper are necessary, Wainwright raises the question of obtaining a converse result by comparing to an oracle method that directly minimizes the rank and the cardinality of the support of the components. A difficulty with this approach is that we don't have a good handle on the set of matrices that are expressible as the sum of a sparse matrix and a low-rank matrix. The properties of this set remain poorly understood, and developing a better picture has been the focus of research efforts in algebraic geometry [6] and in complexity theory [14]. Nonetheless, a comparison to oracle estimators that have side information about the support of the sparse component and the row/column spaces of the low-rank component (in effect, side information about the tangent spaces at the two components) appears to be more tractable. This is closer to the viewpoint we have taken in our paper in which we consider the question of identifiability of the components given information about the underlying tangent spaces. Essentially, our Fisher information conditions state that these tangent spaces must be sufficiently transverse with respect to certain natural norms and in a space in which the

Fisher information is the underlying inner-product. More generally, as also pointed out by Giraud and Tsybakov, the necessity of Fisher information based conditions is an open question even in the sparse graphical model selection setting considered in [11]. The experimental studies in [11] describing comparisons to neighborhood selection in some simple cases provide a good starting point.

Wainwright raises the broader question of consistent model selection when transversality of the underlying tangent spaces does not hold. One approach [1] is to quantify the level of identifiability based on a "spikiness" condition. A more geometric viewpoint may be that only those pieces of the sparse and low-rank components that do not lie in the intersection of their underlying tangent spaces are fundamentally identifiable and, therefore, consistency should be quantified with respect to these identifiable pieces.

Giraud and Tsybakov ask about the interpretability of our conditions $\xi(T)$ and $\mu(\Omega)$. These quantities are geometric in nature and relate to the tangent space conditions for identifiability. In particular, they are closely related to (and bounded by) the incoherence of the row/column spaces of the low-rank component and the maximum number of nonzeros per row/column [5]. These latter quantities have appeared in many papers on sparse graphical model selection (e.g., [9, 11]) as well as on low-rank matrix completion [3], and computing them is straightforward. In our previous work on matrix decomposition [5], we note that these quantities are bounded for natural random families of sparse and low-rank matrices based on results in [3].

5. Experimental issues and applications. Lauritzen and Meinshausen as well as Giraud and Tsybakov raise several points about the choice of the regularization parameters. Choosing these parameters in a data-driven manner (e.g., using the methods described in [10]) is clearly desirable. We do wish to emphasize that the sensitivities of the solution with respect to the parameters λ_n and γ are qualitatively different. As described in our main theorem and in our experimental section, the solution of our estimator (1.1) is stable for a range of values of γ (see also [5])—this point is observed by Yuan as well in his experiments. Further, the choice of γ ideally should not depend on n, while the choice of λ_n clearly should.

On a different point regarding experimental results, Giraud and Tsybakov suggest at the end of their discussion that latent variable models don't seem to provide significantly more expressive power than a sparse graphical model. In contrast, Yuan's synthetic experiment seems to provide compelling evidence that our approach (1.1) provides better performance relative to models learned by the graphical lasso. The reason for these different observations may be tied to the manner in which their synthetic models were generated. Specifically, latent variable model selection using (1.1) is likely to be most useful when the latent variables affect many observed variables upon

marginalization (e.g., latent variables are connected to many observed variables), while the conditional graphical model among the observed variables conditioned on the latent variables is sparse and has bounded degree. This intuition is based on the theoretical analysis in our paper and is also the setting considered in the experiment in Yuan's discussion (as well as in the synthetic experiments in our paper). On the other hand, the experimental setup followed by Giraud and Tsybakov seems to generate a graphical model with large maximum degree and low average degree, and randomly selects a subset of the variables as latent variables. It is not clear if these latent variables are the ones with large degree, which may explain their remarks.

Finally, we note that sparse and low-rank matrix decomposition is relevant in applications beyond the one described in our paper. As observed by Lauritzen and Meinshausen, a natural matrix decomposition problem involving *covariance* matrices may arise if one considers directed latent variable models in the spirit of factor analysis. In such a context the covariance matrix may be expressed as the sum of a low-rank matrix and a sparse (rather than just diagonal) matrix, corresponding to the setting in which the distribution of the observed variables conditioned on the latent variables is given by a sparse covariance matrix. More broadly, similar matrix decomposition problems arise in domains beyond statistical estimation such as optical system decomposition, matrix rigidity and system identification in control [5], as well as others as noted by Candès and Soltanolkotabi.

6. Future questions. Our paper and the subsequent discussions raise a number of research and computational challenges in latent variable modeling that we wish to highlight briefly.

6.1. *Convex optimization in R.* As mentioned by Lauritzen and Meinshausen, R remains the software of choice for practitioners in statistics. However, some of the recent advances in high-dimensional statistical estimation have been driven by sophisticated convex optimization based procedures that are typically prototyped using packages such as SDPT3 [13] and others in Matlab and Python. It would be of general interest to develop packages to invoke SDPT3 routines directly from R.

6.2. *Sparse/low-rank decomposition as infimal convolution.* Given a matrix $M \succ 0$, consider the following function:

$$\|M\|_{S/L,\gamma} = \min_{S,L} \gamma\|S\|_{\ell_1} + \operatorname{tr}(L), \qquad \text{s.t. } M = S - L, L \succeq 0. \tag{6.1}$$

It is clear that $\|\cdot\|_{S/L,\gamma}$ is a norm, and it can be viewed as the infimal convolution [12] of the (scaled) ℓ_1 norm and the trace norm. In essence, it is a norm whose minimization induces matrices expressible as the sum of sparse and low-rank components (see also the atomic norm viewpoint of [4]).

We could then effectively restate (1.1) as

$$\hat{M}_n = \arg\min_{M \succ 0} -\ell(M; \Sigma_O^n) + \lambda_n \|M\|_{S/L,\gamma}$$

and then decompose $\hat{M}_n$ by solving (6.1). This two-step approach suggests the possibility of decoupling the decomposition problem from the conditions fundamentally required for consistency via regularized maximum-likelihood, as the latter only ought to depend on the composite norm $\|\cdot\|_{S/L,\gamma}$. This decoupling also highlights the different roles played by the parameters λ_n and γ (as discussed in Section 5). More broadly, such an approach may be useful as one analyzes general regularizers, for example, convex penalties other than the trace norm as described in Section 3.

6.3. *Non-Gaussian latent variable modeling.* As described in our paper and as raised by Wainwright, latent variable modeling with non-Gaussian variables is of interest in many applications. Both the computational and algebraic aspects present major challenges in this setting. Specifically, the secant varieties arising due to marginalization in non-Gaussian models (e.g., in models with categorical variables) are poorly understood, and computing the likelihood is also intractable. An approach based on matrix decomposition as described in our paper may be appropriate, although one would have to quantify the effects of the Gaussianity assumption.

REFERENCES

[1] Agarwal, A., Negahban, S. and Wainwright, M. J. (2011). Noisy matrix decomposition via convex relaxation: Optimal rates in high dimensions. Preprint.

[2] Candes, E. and Tao, T. (2007). The Dantzig selector: Statistical estimation when p is much larger than n. *Ann. Statist.* **35** 2313–2351. MR2382644

[3] Candès, E. J. and Recht, B. (2009). Exact matrix completion via convex optimization. *Found. Comput. Math.* **9** 717–772. MR2565240

[4] Chandrasekaran, V., Recht, B., Parrilo, P. A. and Willsky, A. S. (2010). The convex geometry of linear inverse problems. Preprint.

[5] Chandrasekaran, V., Sanghavi, S., Parrilo, P. A. and Willsky, A. S. (2011). Rank-sparsity incoherence for matrix decomposition. *SIAM J. Optim.* **21** 572–596. MR2817479

[6] Drton, M., Sturmfels, B. and Sullivant, S. (2007). Algebraic factor analysis: Tetrads, pentads and beyond. *Probab. Theory Related Fields* **138** 463–493. MR2299716

[7] Loh, P. and Wainwright, M. J. (2011). High-dimensional regression with noisy and missing data: Provable guarantees with non-convexity. Preprint.

[8] Ma, S., Xue, L. and Zou, H. (2012). Alternating direction methods for latent variable Gaussian graphical model selection. Technical report, IMA and School of Statistics, Univ. Minnesota.

[9] Meinshausen, N. and Bühlmann, P. (2006). High-dimensional graphs and variable selection with the lasso. *Ann. Statist.* **34** 1436–1462. MR2278363

[10] Meinshausen, N. and Bühlmann, P. (2010). Stability selection. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **72** 417–473. MR2758523

[11] Ravikumar, P., Wainwright, M. J., Raskutti, G. and Yu, B. (2011). High-dimensional covariance estimation by minimizing ℓ_1-penalized log-determinant divergence. *Electron. J. Stat.* **5** 935–980. MR2836766

[12] Rockafellar, R. T. (1997). *Convex Analysis*. Princeton Univ. Press, Princeton, NJ. MR1451876

[13] Toh, K. C., Todd, M. J. and Tutuncu, R. H. SDPT3—A MATLAB software package for semidefinite-quadratic-linear programming. Available at `http://www.math.nus.edu.sg/~mattohkc/sdpt3.html`.

[14] Valiant, L. G. (1977). Graph-theoretic arguments in low-level complexity. In *Mathematical Foundations of Computer Science (Proc. Sixth Sympos., Tatranská Lomnica, 1977). Lecture Notes in Comput. Sci.* **53** 162–176. Springer, Berlin. MR0660702

[15] Wang, C., Sun, D. and Toh, K.-C. (2010). Solving log-determinant optimization problems by a Newton-CG primal proximal point algorithm. *SIAM J. Optim.* **20** 2994–3013. MR2735941

[16] Zhao, P. and Yu, B. (2006). On model selection consistency of Lasso. *J. Mach. Learn. Res.* **7** 2541–2563. MR2274449

V. Chandrasekaran
Department of Computing
and Mathematical Sciences
California Institute of Technology
Pasadena, California 91106
USA
E-mail: venkatc@caltech.edu

P. A. Parrilo
A. S. Willsky
Laboratory for Information
and Decision Systems
Department of Electrical Engineering
and Computer Science
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139
USA
E-mail: parrilo@mit.edu
willsky@mit.edu